Efficient numerical evaluation of Landau coefficients in weakly non-linear stability analysis

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We present an efficient numerical method for evaluating Landau coefficients, which describe small-amplitude equilibrium states in the vicinity of the linear stability threshold. The method differs from the standard approach by the application of solvability condition to the discretized rather than continuous problem. Thus we avoid both the solution of the adjoint problem and the subsequent evaluation of the integrals defining the inner products in the standard approach. Instead of the adjoint eigenfunction we use the left eigenvector of the discretized problem. The latter is supplied by the linear stability analysis together with the right eigenvector for the critical perturbation. Expanding equilibrium solution in small perturbation amplitude in the vicinity of the linear stability threshold, we obtain a matrix eigenvalue perturbation problem. Solvability of this problem requires its inhomogeneous term to be orthogonal to the left eigenvector. The method is demonstrated by using a Chebyshev collocation method to reproduce Landau coefficients for plane Poiseuille flow.

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I. INTRODUCTION

A number of fluid flows can become turbulent while being linearly stable. Some of these flows, such as, for example, plane Couette flow and circular pipe (Hagen-Poiseuille) flow, are linearly stable at all velocities, while some other may become linearly unstable at higher velocities. A typical example of the latter class of flows, which will be our main concern in this study, is plane Poiseuille flow. Theoretically, this flow is known to be linearly stable up to the critical Reynolds number $Re_c = 5722.22$,[1] however, experimentally it has been observed to become turbulent at Reynolds numbers as low as 1000.[2–4] Moreover, turbulence is observed to develop in this flow on the time scales which are by several orders of magnitude shorter than those predicted by the linear stability analysis.[5] Such a subcritical instability can be accounted for by the positive feedback of perturbation on its growth rate, which is a non-linear effect beyond the scope of the linear stability theory. Thus, a perturbation of sufficiently large amplitude can acquire a large positive growth rate at subcritical Reynolds numbers, where all small-amplitude perturbations are linearly stable. For small-amplitude perturbations in the vicinity of linear stability threshold, this type of phenomenon is described, in general, by the so-called Landau equation.[6,7] Whether instability is sub- or supercritical is determined by the coefficients of this equation, which are referred to as Landau coefficients and have to be determined for each particular case.

Basic formalism for deriving Landau coefficients for plane Poiseuille flow has been developed by Stuart[8] and Watson.[9] A more up-to-date account of this approach is given by Schmid and Henningson.[10] This method has been extended and modified by Reynolds and Potter[11] who used it to prove that plane Poiseuille flow is indeed subcritically unstable. Both aforementioned methods were compared by Sen and Venkateswarlu[12] who applied them to calculate higher-order Landau coefficients for plane Poiseuille flow driven by a fixed pressure gradient. Not much difference between the methods was detected in the supercritical region, however the Watson method was found difficult to apply in the subcritical region, which represents the main interest for this flow. These type of asymptotic expansion methods have been reconsidered and surveyed by Herbert,[13] and substantially extended by Stewartson and Stuart[14] who included a slow spatial variation, which resulted in the complex Ginzburg-Landau equation.[15]

The evaluation of the Landau coefficients required in weakly non-linear stability analy-

sis is technically rather complicated. This may explain why most hydrodynamic stability stability problems are restricted to the linear analysis, which is of a limited practical significance when the instability happens to be subcritical. A significant technical hindrance to the implementation of weakly non-linear stability analysis is the adjoint eigenfunction that needs to be found by solving the adjoint problem. Then several complex inner product integrals containing the adjoint eigenfunction need to be evaluated in order to obtain Landau coefficients.

In this paper, we develop a simpler but numerically more accurate method for evaluating Landau coefficients. The method is based on the application of solvability condition to the discretized rather than continuous problem. This allows us to evaluate Landau coefficients without using the adjoint eigenfunction, which in our approach is replaced by the left eigenvector. Such a possibility has been briefly discussed by Crouch and Herbert.[16] A similar approach employing Gaussian elimination has been noted earlier by Sen and Venkateswarlu.[12]

The rest of the paper is organized as follows. In the section below, we formulate the problem for plane Poiseuille flow and introduce 2D traveling-wave solution. The latter is then expanded in small perturbation amplitude in the vicinity of linear stability threshold to re-derive the classic expressions for Landau coefficients. Section III presents a detailed development of our approach for a Chebyshev collocation method, which is used to evaluate Landau coefficients for plane Poiseuille flow at its linear stability threshold. The paper is concluded by a summary of results in Sec. IV.

II. FORMULATION OF PROBLEM

Consider a flow of incompressible liquid with density ρ and kinematic viscosity ν driven by a constant pressure gradient $\boldsymbol{\nabla} p_0 = -\mathbf{e}_x P_0$ in the gap between two parallel walls located $z = \pm h$ in Cartesian system of coordinates with the x and z axes directed streamwise and transverse to the walls, respectively. The velocity distribution $\mathbf{v}(\mathbf{r}, t)$ is governed by the Navier-Stokes equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\rho^{-1} \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{v} \tag{1}$$

and subject to the incompressiblity constraint $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. Subsequently, all variables are non-dimensionalized by using h and h^2/ν as the length and time scales, respectively. In order to simplify the expressions of Landau coefficients obtained in the following, we employ

the viscous diffusion speed ν/h as the characteristic velocity instead of the commonly used maximum flow velocity. Due to this non-standard scaling Reynolds number appears as a factor at the convective term rather reciprocal at the viscous term. The problem admits a rectilinear base flow

$$\mathbf{v}_0(z) = Re\bar{u}(z)\mathbf{e}_x, \tag{2}$$

where $\bar{u}(z) = 1 - z^2$ is the parabolic velocity profile and $Re = U_0h/\nu$ is the Reynolds number based on the centerline velocity $U_0 = 2P_0h^2/\rho\nu$. At sufficiently high Re, the base flow can become unstable with respect to infinitesimal perturbations $\mathbf{v}_1(\mathbf{r}, t)$, which due to the invariance of the base flow in both t and $\mathbf{x} = (x, y)$ can be sought as

$$\mathbf{v}_1(\mathbf{r}, t) = \hat{\mathbf{v}}_1(z)\mathrm{e}^{\lambda t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}, \tag{3}$$

where λ is the complex temporal growth rate for the Fourier mode with the wave vector $\mathbf{k} = (\alpha, \beta)$ and complex amplitude distribution $\hat{\mathbf{v}}_1(z)$. The linear stability analysis aims to find marginal values of Re depending on $\mathbf{k}$ for which neutrally stable perturbations defined by $\Re[\lambda] = 0$ are possible. The lowest marginal value of Re, which is referred to as the critical Reynolds number, for this flow is $Re_c = 5772.22$. It is attained at the critical wave vector $\mathbf{k}_c = (1.02055, 0)$, which corresponds to purely transverse perturbations defined by $\beta = 0$. For $Re > Re_c$, the linear stability theory predicts exponentially growing perturbations. Evolution of these unstable perturbations depends on the nonlinear effects which may either inhibit or enhance the growth rate leading to what is known as super- and subcritical instabilities, respectively. The former is expected to set in only at supercritical Reynolds numbers, while the latter can be triggered by sufficiently large amplitude perturbations also in a certain range of supercritical Reynolds numbers.

A. 2D equilibrium states

In order to determine whether instability is super- or subcritical, we employ the approach of Reynolds and Potter,[11] known as the method of "false problems",[13,17] and search for an equilibrium solution in the vicinity of Re_c as follows. The neutrally stable mode (3) with a purely real frequency $\omega = -\mathrm{i}\lambda$ interacting with itself through the quadratically nonlinear term in Eq. (1) is expected to produce a steady streamwise-invariant perturbation of the mean flow as well as a second harmonic $\sim \mathrm{e}^{2\mathrm{i}(\omega t + \alpha x)}$. Subsequent nonlinear interactions

are expected to produce higher harmonics, which similarly to the fundamental and second harmonic travel with the same phase velocity $c = -\omega/\alpha$. Thus, the solution can be sought in the form

$$\mathbf{v}(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} E^n \hat{\mathbf{v}}_n(z), \tag{4}$$

where $E = \mathrm{e}^{\mathrm{i}(\omega t + \alpha x)}$ contains ω, which, in general, needs to determined together $\hat{\mathbf{v}}_n$ by solving a non-linear eigenvalue problem. The reality of solution requires $\hat{\mathbf{v}}_{-n} = \hat{\mathbf{v}}_n^*$, where the asterisk stands for the complex conjugate. The incompressiblity constraint applied to the nth velocity harmonic results in $\mathbf{D}_n \cdot \hat{\mathbf{v}}_n = 0$, where $\mathbf{D}_n \equiv \mathbf{e}_z \frac{d}{dz} + \mathrm{i}\mathbf{e}_x \alpha_n$ with $\alpha_n = \alpha n$ stands for the spectral counterpart of the nabla operator. This constraint can be satisfied by expressing the streamwise velocity component

$$\hat{u}_n = \mathbf{e}_x \cdot \hat{\mathbf{v}}_n = \mathrm{i}\alpha_n^{-1} \hat{w}_n' \tag{5}$$

in terms of the transverse component $\hat{w}_n = \mathbf{e}_z \cdot \hat{\mathbf{v}}_n$, which we employ instead of the commonly used stream function. Henceforth the prime is used as a shorthand for d/dz. Note that Eq. (5) is not applicable to the zeroth harmonic, for which it yields $\hat{w}_0 \equiv 0$. Thus, $\hat{u}_0$ needs to be considered separately in our velocity-based approach.

Taking the *curl* of Eq. (1) to eliminate the pressure gradient and then projecting it onto $\mathbf{e}_y$ we obtain

$$[\mathbf{D}_n^2 - \mathrm{i}\omega n]\hat{\zeta}_n = \hat{h}_n, \tag{6}$$

where

$$\hat{\zeta}_n = \mathbf{e}_y \cdot \mathbf{D}_n \times \hat{\mathbf{v}}_n = \begin{cases} \mathrm{i}\alpha_n^{-1} \mathbf{D}_n^2 \hat{w}_n, & n \neq 0; \\ \hat{u}_0', & n = 0. \end{cases} \tag{7}$$

and

$$\hat{h}_n = \sum_m \hat{\mathbf{v}}_{n-m} \cdot \mathbf{D}_m \hat{\zeta}_m \tag{8}$$

are the y-components of the nth harmonic of the vorticity $\boldsymbol{\zeta} = \boldsymbol{\nabla} \times \mathbf{v}$ and that of the *curl* of the non-linear term $\mathbf{h} = \boldsymbol{\nabla} \times (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$. Henceforth, the omitted summation limits are assumed to be infinite. Separating the terms involving $\hat{u}_0$ in sum (8), it can be rewritten as $\hat{h}_n = \mathrm{i}\alpha_n^{-1}(\hat{h}_n^w + \hat{h}_n^u)$, where

$$\hat{h}_n^w = n \sum_{m \neq 0} m^{-1} (\hat{w}_{n-m} \mathbf{D}_n^2 \hat{w}_n' - \hat{w}_m' \mathbf{D}_{n-m}^2 \hat{w}_{n-m}), \tag{9}$$

$$\hat{h}_n^u = \mathrm{i}\alpha_n [\hat{u}_0 - \hat{u}_0'' \mathbf{D}_n^2] \hat{w}_n = \mathcal{N}_n(\hat{u}_0) \hat{w}_n. \tag{10}$$

Eventually, using the expressions above, Eq. (6) can be written as

$$\mathcal{L}_n(\mathrm{i}\omega, \hat{u}_0)\hat{w}_n = \hat{h}_n^w, \tag{11}$$

with the operator

$$\mathcal{L}_n(\mathrm{i}\omega, \hat{u}_0) = [\mathbf{D}_n^2 - \mathrm{i}\omega n]\mathbf{D}_n^2 - \mathcal{N}_n(\hat{u}_0). \tag{12}$$

The equation above governs all harmonics except the zeroth one, for which it implies $\hat{w}_0 \equiv 0$ in accordance with the incompressiblity constraint (5). The zeroth velocity harmonic, which has only the streamwise component $\hat{u}_0$, is governed directly the x-component of Navier-Stokes equation (1):

$$\hat{u}_0'' = -\hat{P}_0 + \hat{g}_0, \tag{13}$$

where $\hat{P}_0$ is the dimensionless mean pressure gradient and

$$\hat{g}_0 = \mathrm{i}\sum_{m\neq 0} \alpha_m^{-1}\hat{w}_m^*\hat{w}_m'' \tag{14}$$

is the x-component of the zeroth harmonic of the nonlinear term $\mathbf{g} = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$. Velocity harmonics obey the usual no-slip and impermeability boundary conditions

$$\hat{w}_n = \hat{w}_n' = \hat{u}_0 = 0 \text{ at } z = \pm 1. \tag{15}$$

B. Amplitude expansion

The equations obtained above govern equilibrium states of 2D traveling waves of arbitrary amplitude. In the vicinity of the linear stability threshold, which represents the main interest here, the solution can be simplified by expanding in small perturbation amplitude. As discussed above, the basic harmonic (3) with a small amplitude $O(\epsilon)$ interacting with itself through the quadratically nonlinear term in Eq. (1) produces the zeroth harmonic, which modifies the mean flow, and the second harmonic, both with amplitude $O(\epsilon^2)$. These two harmonics further interacting with the basic one produce an $O(\epsilon^3)$ correction to the latter. The second harmonic interacting with the basic one also gives rise to the third harmonic with amplitude $O(\epsilon^3)$. This perturbation series is represented by the following expansion:

$$\hat{w}_n = \sum_{m=0}^{\infty} \epsilon^{|n|+2m}\tilde{A}^{|n|}|\tilde{A}|^{2m}\hat{w}_{n,|n|+2m}, \tag{16}$$

where $\epsilon\tilde{A} = A$ is an unknown equilibrium amplitude of the basic harmonic and $\tilde{A} = O(1)$ is its normalized counterpart. The mean flow, which as mentioned above needs to be considered separately, is expanded respectively as

$$\hat{u}_0 = \hat{u}_{0,0} + \epsilon^2|\tilde{A}|^2\hat{u}_{0,2} + \dots. \tag{17}$$

Similarly, we expand also the Reynolds number and frequency

$$Re = Re_0 + \epsilon^2\tilde{Re}_2 + \dots, \tag{18}$$

$$\omega = \omega_0 + \epsilon^2\tilde{\omega}_2 + \dots, \tag{19}$$

where Re_0 is a marginal Reynolds number satisfying $\Re[\lambda_0] = 0$ for the mode $\hat{w}_{1,1}$ with the frequency $\omega_0 = \Im[\lambda_0]$ and the wavenumber α; $\epsilon^2\tilde{Re}_2 = Re_2$ and $\epsilon^2\tilde{\omega}_2 = \omega_2$ are the deviations of Reynolds number and frequency from their linear stability threshold values. Substituting these expansions into Eqs. (11) and (13), and collecting the terms at equal powers of ϵ, we obtain the following equations. At $O(\epsilon^0)$, we have the base flow equation

$$\hat{u}''_{0,0} = -P_{0,0}, \tag{20}$$

where $P_{0,0} = 2Re_0$ and $\hat{u}_{0,0} = Re_0(1-z^2) = Re_0\bar{u}(z)$. At $O(\epsilon)$, we recover the Orr-Sommerfeld equation

$$\mathcal{L}_1(\mathrm{i}\omega_0, \hat{u}_{0,0})\hat{w}_{1,1} = 0, \tag{21}$$

which defines the linear stability threshold. Solution of this eigenvalue problem for a given wavenumber α yields Re_0, ω_0, $\hat{w}_{1,1}(z)$. The latter is defined up to an arbitrary factor, which for $\hat{w}_{1,1}(z)$ being an even function is determined by the standard normalization condition

$$\hat{w}_{1,1}(0) = 1. \tag{22}$$

At $O(\epsilon^2)$, two equations are obtained

$$\hat{u}''_{0,2} = -P_{0,2} - 2\alpha^{-1}\Im[\hat{w}^*_{1,1}\hat{w}''_{1,1}], \tag{23}$$

$$\mathcal{L}_2(\mathrm{i}\omega_0, \hat{u}_{0,0})\hat{w}_{2,2} = 2[(\hat{w}_{1,1}\hat{w}'_{1,1})' - 2\hat{w}'^2_{1,1}]', \tag{24}$$

which define the mean-flow perturbation $\hat{u}_{0,2}$ and the second harmonic $\hat{w}_{2,2}$ in terms of $\hat{w}_{1,1}(z)$. The mean-flow perturbation depends also on the perturbation on the mean pressure gradient $P_{0,2}$, which is zero when the flow is driven by a fixed pressure difference. Alternatively, if it is the flow rate rather than the pressure difference which is fixed, then $P_{0,2}$ is an

additional unknown which has be determined by using the flow-rate conservation condition $\int_{-1}^{1} \hat{u}_{0,2}(z)\,dz = 0$. We start with fixed mean pressure gradient $P_{0,2} = 0$. The the fixed flow rate case can readily be reduced to the former by including $P_{0,2}$ into Re_2 as shown later on.

To complete the solution, we need to proceed to the order $O(\epsilon^3)$, which yields

$$\mathcal{L}_1(\mathrm{i}\omega_0, \hat{u}_{0,0})\hat{w}_{1,3} = \hat{h}^w_{1,3} + |A|^{-2}[\mathcal{N}_1(Re_2\bar{u} + |A|^2\hat{u}_{0,2}) + \mathrm{i}\omega_2\mathbf{D}_1^2]\hat{w}_{1,1}, \tag{25}$$

where

$$\hat{h}^w_{1,3} = \frac{1}{2}\left(\hat{w}^*_{1,1}\mathbf{D}_2^2\hat{w}'_{2,2} - \hat{w}'_{2,2}\mathbf{D}_1^2\hat{w}^*_{1,1}\right) - \left(\hat{w}_{2,2}\mathbf{D}_1^2\hat{w}'^*_{1,1} - \hat{w}'^*_{1,1}\mathbf{D}_2^2\hat{w}_{2,2}\right). \tag{26}$$

Equation (25) defines the correction of the basic harmonic $\hat{w}_{1,3}$ in terms of the lower harmonic perturbations described above. It is important to notice that the l.h.s. operator of Eq. (25) is the same as that of the homogenous Eq. (21), which is satisfied by $\hat{w}_{1,1}$. Thus, for Eq. (25) to be solvable, its r.h.s. cannot contain term proportional to $\hat{w}_{1,1}$. Namely, the r.h.s has to be orthogonal to the adjoint eigenfunction $\hat{w}^+_{1,1}$:

$$\left\langle \hat{w}^+_{1,1}, \hat{h}^w_{1,3} + |A|^{-2}[\mathcal{N}_1(Re_2\bar{u} + |A|^2\hat{u}_{0,2}) + \mathrm{i}\omega_2\mathbf{D}_1^2]\hat{w}_{1,1} \right\rangle = 0, \tag{27}$$

where the angle brackets denote the inner product.[10] This solvability can be rewritten as

$$\mathrm{i}\omega_2 = \mu_1 Re_2 + \mu_2|A|^2, \tag{28}$$

where

$$\mu_1 = -\left\langle \hat{w}^+_{1,1}, \mathcal{N}_1(\bar{u})\hat{w}_{1,1} \right\rangle, \tag{29}$$

$$\mu_2 = -\left\langle \hat{w}^+_{1,1}, \mathcal{N}_1(\hat{u}_{0,2})\hat{w}_{1,1} + \hat{h}^w_{1,3} \right\rangle \tag{30}$$

for the adjoint eigenfunction normalized as $\left\langle \hat{w}^+_{1,1}, \mathbf{D}_1^2\hat{w}_{1,1} \right\rangle = 1$. Equation (28) represents a reduced Landau equation for the case of equilibrium solution, which requires ω_2 to be real. This reality condition yields the sought equilibrium amplitude

$$|A|^2 = -Re_2\Re[\mu_1]/\Re[\mu_2], \tag{31}$$

which is the same as that resulting from the full Landau equation with the first Landau coefficient μ_2 and the linear growth rate correction $\mu_1 Re_2$.[10] Note that our non-standard choice of the characteristic velocity results in expressions (29) and (30) sharing operator $\mathcal{N}_1$, which simplifies their numerical evaluation in the next section.

The type of instability is determined by the sign of $\Re[\mu_2]$. For an instability to be supercritical, which supposes an equilibrium solution with $|A|^2 > 0$ to exist at positive linear growth rates $Re_2\Re[\mu_1] > 0$, $\Re[\mu_2] < 0$ is required. Otherwise, instability is subcritical. In order to calculate Landau coefficients (29) and (30) following the standard approach outlined above one needs to solve not only the Orr-Sommerfeld equation (21) but also its adjoint problem for $\hat{w}^+_{1,1}$. Both the direct and adjoint problems, as well as those posed by Eqs. (23) and (24), need to be solved numerically. Then the integrals in the inner products defining μ_1 and μ_2 need to be evaluated numerically. This standard approach can be significantly simplified by avoiding both the solution of the adjoint problem and the evolution of the inner product integrals. This is achieved by applying the solvability condition to the discretized problem as demonstrated in the following.

III. NUMERICAL SOLUTION

In this section, numerical evaluation of Landau coefficients will be demonstrated using a Chebyshev collocation method with Chebyshev-Lobatto nodes

$$z_i = \cos(i\pi/N), \quad i = 0, \cdots, N, \tag{32}$$

at which the discretized solution $(\hat{w}_n, \hat{u}_0)(z_i) = (\tilde{w}_n, \tilde{u}_0)_i$ and its derivatives are sought. The latter are expressed in terms of the former by using the so-called differentiation matrices which for the first and second derivatives are denoted by $D^{(1)}_{i,j}$ and $D^{(2)}_{i,j}$. Explicit expressions of these matrices, which are too long to presented here, are given by Peyret.[18] Equations (21), (23) and (24) are approximated at the internal collocation points $0 < i < N$ and the boundary conditions (15) are imposed at the boundary points $i = 0, N$. The operator $\mathcal{L}_n(\mathrm{i}\omega_0, \hat{u}_{0,0})$ defined by Eq. (12), which appears in Eqs. (21) and (24) is represented by the matrix

$$\mathbf{L}_n(\mathrm{i}\omega_0, \mathbf{u}_{0,0}) = \mathbf{M}_n(\mathbf{u}_{0,0}) - \mathrm{i}\omega_0\mathbf{A}_n,$$

which contains

$$\mathbf{M}_n(\mathbf{u}_{0,0}) = \mathbf{F}_n[\mathbf{A}_n^2 + Re_0\mathbf{N}_n(\bar{\mathbf{u}})], \tag{33}$$

$$(\mathbf{A}_n)_{i,j} = (\mathbf{D}_n^2)_{i,j}, \quad 0 < (i,j) < N, \tag{34}$$

where the latter represents the part of the collocation approximation of the operator

$$(\mathbf{D}_n^2)_{i,j} = D^{(2)}_{i,j} - \alpha_n^2\delta_{i,j} \tag{35}$$

related with the internal nodes. The other matrix in Eq. (33),

$$(\mathbf{N}_n(\bar{\mathbf{u}}))_{i,j} = \mathrm{i}\alpha_n[\bar{u}_i\delta_{i,j} - \bar{u}_i''(\mathbf{A}_n)_{i,j}], \tag{36}$$

represents the collocation of approximation the operator defined by Eq. (10). Finally, the factor matrix[19]

$$\mathbf{F}_n = \mathbf{I} - \mathbf{B}_n(\mathbf{C}\mathbf{A}_n^{-1}\mathbf{B}_n)^{-1}\mathbf{C}\mathbf{A}_n^{-1} \tag{37}$$

in Eq. (33) is due to the no-slip boundary condition $\hat{w}'(\pm 1) = 0$, which is represented by $\mathbf{C}\mathbf{w} = \mathbf{0}$ with

$$C_{ij} = D_{i,j}^{(1)}, \quad i = 0, N; 0 < j < N. \tag{38}$$

It also involves the part of the operator (35) related with the boundary nodes:

$$(\mathbf{B}_n)_{i,j} = (\mathbf{D}_n^2)_{i,j}, \quad 0 < i < N, j = 0, N. \tag{39}$$

We start with the Orr-Sommerfeld equation, whose collocation approximation

$$\mathbf{L}_1(\lambda, Re\bar{\mathbf{u}})\mathbf{w}_{1,1} = [\mathbf{M}_1(Re\bar{\mathbf{u}}) - \lambda\mathbf{A}_1]\mathbf{w}_{1,1} = \mathbf{0} \tag{40}$$

after multiplication by $\mathbf{A}_1^{-1}$ reduces to the standard complex matrix eigenvalue problem

$$[\mathbf{A}_1^{-1}\mathbf{M}_1(Re\bar{\mathbf{u}}) - \lambda\mathbf{I}]\mathbf{w}_{1,1} = \mathbf{0}, \tag{41}$$

which can be solved using, for example, LAPACK's ZGEEV routine.[20] The marginal Reynolds number Re_0 for a given wavenumber α is determined by the condition $\Re[\lambda_0] = 0$ for the eigenvalue λ_0 with the largest real part. Simultaneously with the right eigenvector $\mathbf{w}_{1,1}$, we find also the associated left eigenvector $\mathbf{w}_{1,1}^{\dagger}$.[21] The right eigenvector is normalized using condition (22), and the left one is normalized against the former using the dot product of complex vectors $\mathbf{w}_{1,1}^{\dagger} \cdot \mathbf{w}_{1,1} = 1$. This normalization simplifies the expressions of Landau coefficients obtained below. Having found $\mathbf{w}_{1,1}$, we can straightforwardly solve discretized counterparts of Eqs. (23) and (24), which yield the mean-flow perturbation $\mathbf{u}_{0,2}$ and the complex amplitude distribution of the second harmonic $\mathbf{w}_{2,2}$. For the fixed flow rate considered later on, we need also the stream function of the mean-flow perturbation $\boldsymbol{\psi}_{0,2}$, which is obtained by solving the collocation approximation of $\hat{\psi}_{0,2}' = \hat{u}_{0,2}$ with the symmetry condition $\hat{\psi}_{0,2}(0) = 0$.

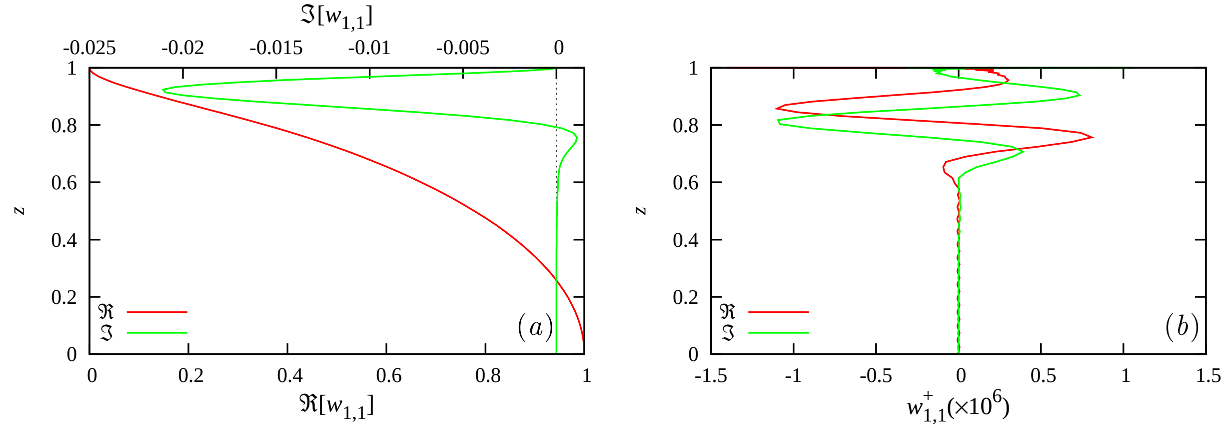

Figure 1. Real and imaginary parts of the critical perturbation $\hat{w}_{1,1}$ given by the right eigenvector $\mathbf{w}_{1,1}$ (a) and those of the respective left eigenvector $\mathbf{w}^{\dagger}_{1,1}$ (b).

Now we can proceed to solving the final equation (25), whose collocation approximation can be written similarly to Eq. (40) as

$$\mathbf{L}_1(\mathrm{i}\omega_0, Re_0\bar{\mathbf{u}})\mathbf{w}_{1,3} = \mathbf{F}_1\mathbf{h}^w_{1,3} + |A|^{-2}[\mathbf{F}_1\mathbf{N}_1(Re_2\bar{\mathbf{u}} + |A|^2\mathbf{u}_{0,2}) + \mathrm{i}\omega_2\mathbf{A}_1]\mathbf{w}_{1,1}, \tag{42}$$

which represents a matrix eigenvalue perturbation problem. For this system of linear equation to be solvable, its r.h.s multiplied by $\mathbf{A}_1^{-1}$ as in Eq. (41) has to be orthogonal to $\mathbf{w}^{\dagger}_{1,1}$.[22] This discrete solvability condition leads to the same reduced Landau equation (28), whose coefficients now are

$$\mu_1 = -\mathbf{w}^{\dagger}_{1,1} \cdot \mathbf{A}_1^{-1}\mathbf{F}_1\mathbf{N}_1(\bar{\mathbf{u}})\mathbf{w}_{1,1}, \tag{43}$$

$$\mu_2 = -\mathbf{w}^{\dagger}_{1,1} \cdot \mathbf{A}_1^{-1}\mathbf{F}_1(\mathbf{N}_1(\mathbf{u}_{0,2})\mathbf{w}_{1,1} + \mathbf{h}^w_{1,3}). \tag{44}$$

Owing to the symmetry of the problem, both $\hat{w}_{1,1}$ and $\hat{u}_{0,2}$ are even, whereas $\hat{w}_{2,2}$ is an odd function of z. This allows us to search the solution in one half of the layer so reducing the number of required collocation points by half. $M = N/2 = 32$ collocation points in half layer is sufficient to obtain the the critical Reynolds number $Re_c = 5772.22$, frequency $\omega_c = -1555.18$ and wavenumber $\alpha_c = 1.02055$ with six significant figures.

The real and imaginary parts of the critical perturbation $\hat{w}_{1,1}$, which is given by the right eigenvector $\mathbf{w}_{1,1}$ are shown in Fig. 1 together with the respective left eigenvector $\mathbf{w}^{\dagger}_{1,1}$. Note that the latter is orthogonal to all other right eigenvectors but $\mathbf{w}_{1,1}$ and has only a numerical but no physical meaning. Because of different inner product definitions for the continuous

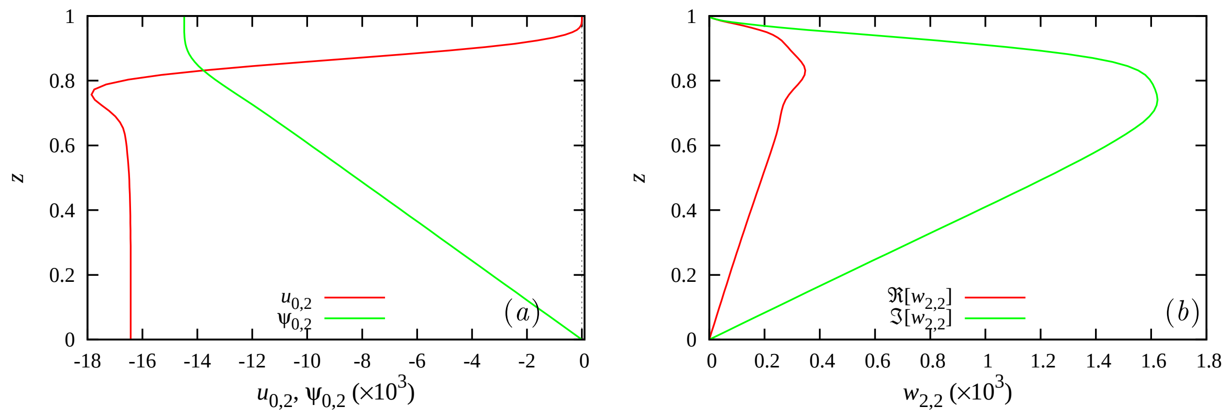

Figure 2. Velocity $\hat{u}_{0,2}$ and the associated stream function $\hat{\psi}_{0,2}$ of the mean-flow perturbation (a); the real and imaginary parts of the second harmonic amplitude $\hat{w}_{2,2}$ (b).

and discrete problems, $\mathbf{w}^{\dagger}_{1,1}$ is also distinct from the adjoint eigenfunction $\hat{w}^{+}_{1,1}$. Distributions of the mean-flow perturbation and that of the complex amplitude of the second harmonic are plotted for the top half of the layer in Fig. 2. Note that due to the non-standard scaling, our dimensionless frequency and velocity differ by a factor of Re_c from the values obtained with the conventional scaling based on the centerline velocity.

Substituting the above results into Eqs. (29) and (30) we obtain

$$\mu_1 = 0.0097118 - \mathrm{i}0.222596,$$
$$\mu_2 = 0.0049382 - \mathrm{i}0.0239131.$$

As seen from Fig. 3, $M \gtrsim 32$ collocation points produce Landau coefficients with about six significant figures. The first and most important result is $\Re[\mu_2] > 0$, which, as discussed above, confirms the subcritical nature of this instability in agreement with the previous studies. The coefficient μ_1 has been computed explicitly by Stewartson and Stuart[14] who found $d_1 = (0.17 + \mathrm{i}0.8) \times 10^{-5}$ for the standard normalization. Rescaling our result with the centerline velocity, we obtain $\tilde{\mu}_1 = \mu_1/Re_c = (0.168251 - \mathrm{i}3.85633) \times 10^{-5}$, whose real part is close to that of d_1, while the imaginary is quite different. The reason for this difference is unclear. In addition, μ_1 can be verified against the numerical results of the linear stability analysis for the complex growth rate in the vicinity of the linear stability threshold, where $\delta\lambda = \lambda - \lambda_c \approx \mu_1(Re - Re_c)$. As seen in Fig. 4, the complex phase speed $c = -\mathrm{i}\lambda/Re\alpha$, which is commonly used instead of λ, is accurately reproduced by μ_1 in the vicinity of Re_c.

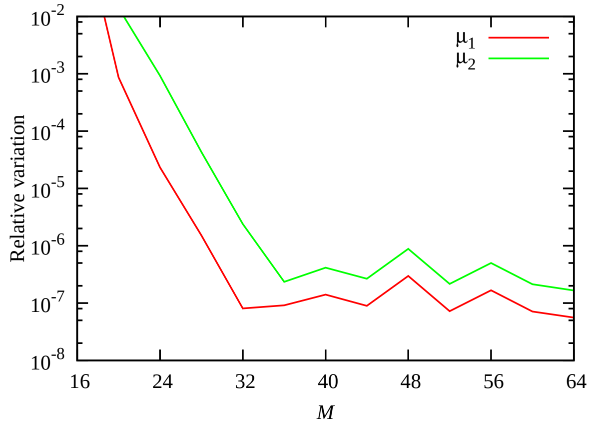

Figure 3. Relative variation of Landau coefficients with the number of collocation points M.

In order to compare our Landau coefficient μ_2 with the values found in the previous studies we have to take into account not only the non-standard scaling but also the fact that in our case A stands for the amplitude of the transverse velocity component w, whereas in the previous studies it denotes the amplitude of the stream function ψ, which are related as $w = \mathrm{i}\alpha\psi$. Thus, our μ_2 rescales as

$$\tilde{\mu}_2 = \mu_2 \alpha_c^2 Re_c = 29.659 - \mathrm{i}143.622.$$

This result is close to $\tilde{\mu}_2 = \mathrm{i}\alpha_c K_1 = 29.46 - \mathrm{i}143.41$ found by Sen and Venkateswarlu[12] using the method of Reynolds and Potter[11] for $Re_c = 5774$, $\alpha_c = 1.02$ and $c_r = 0.2639$. Note that K_1 is mistaken for $\tilde{\mu}_2$ by Schmid and Henningson[10] who denote it by λ_2. Using a Chebyshev collocation method Fujimura[23] found $\Re[\tilde{\mu}_2] = 30.962411$, which appears to be less accurate than the result of Sen and Venkateswarlu[12] obtained by a conventional finite-difference method.

Reynolds and Potter[11] used their original method of "false solution" to obtain the first relatively accurate values of Landau coefficients for the case of fixed flow rate. Our solution for fixed pressure gradient can be converted to the fixed flow rate by using the non-zero pressure gradient correction $P_{0,2}$ in Eq. (23). As seen from Eq. (20), this correction, which affects only the magnitude of the base flow, is equivalent to the substitution of Re_2 by

$$Re_2^q = Re_2 + |A|^2 P_{0,2}/2.$$

The pressure correction $P_{0,2}$, which according to the expression above produces a flow rate perturbation $|A|^2 P_{0,2}\bar{\psi}(1)$, has to compensate flow rate perturbation $2|A|^2\hat{\psi}_{0,2}(1)$, which

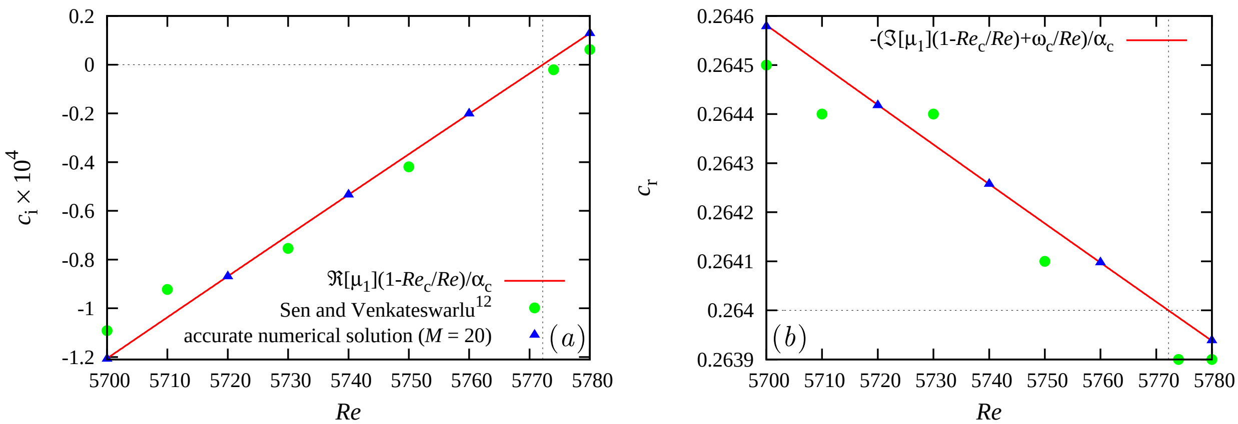

Figure 4. Imaginary (a) and real (b) parts of the complex phase velocity $c = -\mathrm{i}\lambda/Re\alpha$ of the most unstable mode in the vicinity of the critical Reynolds number Re_c calculated using μ_1 and supplied by the linear stability analysis (triangles) and taken from Sen and Venkateswarlu[12] (dots).

occurs at fixed pressure gradient. This results in

$$P_{0,2}/2 = -\hat{\psi}_{0,2}(1)/\bar{\psi}(1) = 0.00217238,$$

where $\bar{\psi}(1) = \int_0^1 \bar{u}(z)\,dz = \frac{2}{3}$. Substitution of Re_2 by Re_2^q in Eq. (28) results in the replacement of μ_2 by

$$\mu_2^q = \mu_2 + \mu_1 P_{0,2}/2 = 0.0051492 - \mathrm{i}0.0287487.$$

Rescaling μ_2^q with the critical Reynolds number based on the mean velocity $\bar{R}e_c = \frac{2}{3}Re_c = 3848.08$ and the critical wavenumber $\alpha_c = 1.02071$ used by Reynolds and Potter,[11] we obtain

$$\tilde{\mu}_2^q = \mu_2^q \alpha_c^2 \bar{R}e_c = 20.64 - \mathrm{i}115.26,$$

which is not far from $\tilde{\mu}_2^q = a^{(2)} + \mathrm{i}b^{(2)} = 19.7 - \mathrm{i}111$ obtained by Reynolds and Potter.[11]

IV. SUMMARY AND CONCLUSION

We have developed an efficient numerical method for evaluating Landau coefficients, which determine weakly non-linear evolution of small but finite amplitude perturbations in the vicinity of linear stability threshold. Our main interest was in the equilibrium states, particularly whether they exist above or below the threshold. This depends on the real part of the first Landau coefficient. If it is positive, the instability is subcritical. Otherwise, the instability is supercritical.

To identify the type of instability we employed the so-called method of "false solution" to search for equilibrium solution using expansion in amplitude, which was supposed to be small in the vicinity of linear stability threshold. This resulted in a matrix eigenvalue perturbation problem, which was solvable only when its inhomogeneous term was orthogonal to the left eigenvector. Application of the solvability condition to the discretized rather than the continuous problem was the main novelty of our approach. It allowed us to avoid both the solution of the adjoint problem and the subsequent evaluation of the integrals defining the inner products in the conventional approach. The adjoint eigenfunction in our approach was substituted by the left eigenvector, whereas the integrals were replaced by the dot products of complex vectors. The left eigenvector was supplied by the solution of the discretized linear stability problem together with the right eigenvector.

The method was demonstrated by using a Chebyshev collocation method to reproduce Landau coefficients for plane Poiseuille flow at its linear stability threshold. Our results agreed well with the published values for both fixed-pressure-gradient and fixed-flow-rate driven flows.

Circumventing solution of the adjoint problem not only considerably simplifies weakly non-linear stability analysis but also increases its accuracy by avoiding approximate evaluation of the inner product integrals. This makes our method more practical than the standard approach.

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